LATTICE BASIS REDUCTION

1. INTRODUCTION TO LATTICES

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Outline

1. Definition of a lattice
2. Bases of a lattice
3. Volume of a lattice
4. Successive minima
5. Gaussian heuristic
6. Shortest Vector Problem (SVP)

Lattice definition

- **Definition**. A *lattice* L in $\mathbb{R}^n$ is the set of all integer linear combinations of m linearly independent vectors $B = [b_1, b_2, \ldots, b_m]$ in $\mathbb{R}^n$ (and where $m \leq n$).
  The set B is called an (ordered) *basis* of L, and we write $L = L(B)$.
  The *dimension* of L is n, and the *rank* of L is m.

- **Notes**:

  1. We will henceforth assume that the basis vectors $b_1, b_2, \ldots, b_m$ are in $\mathbb{Z}^n$.
  2. Thus, $L = \{x_1 b_1 + x_2 b_2 + \cdots + x_m b_m \ : \ x_1, x_2, \ldots, x_m \in \mathbb{Z}\} \subseteq \mathbb{Z}^n$.
     L is called an *integer lattice*.
  3. Let B be the $n \times m$ matrix whose columns are the basis vectors $b_1, \ldots, b_m$

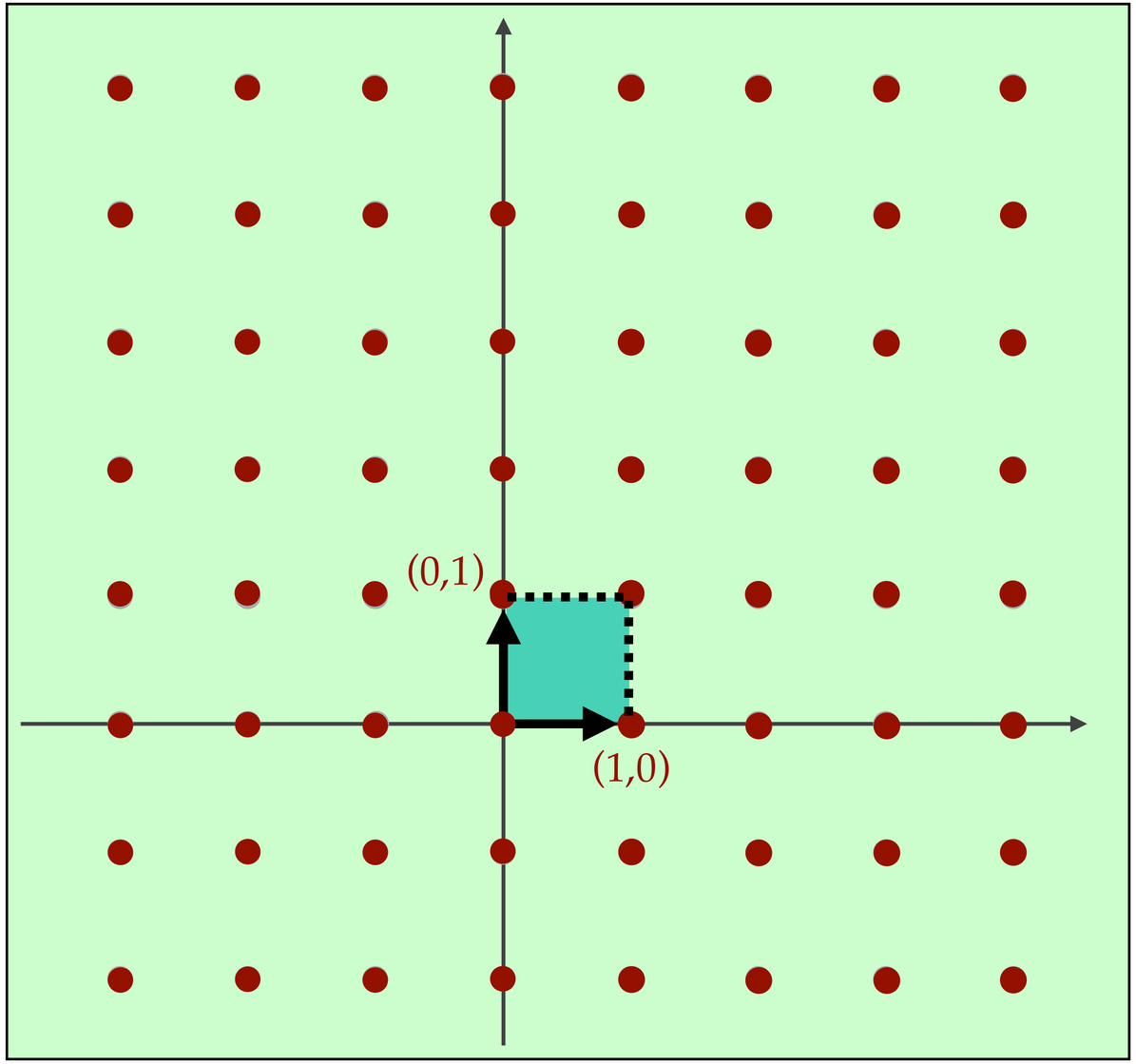
     so $B = \begin{bmatrix} | & | & \cdots & | \\ b_1 & b_2 & \cdots & b_m \\ | & | & \cdots & | \end{bmatrix}$. Then $L = \{Bx : x \in \mathbb{Z}^m\}$.

Full-rank lattices

- **Definition**. Let L and L' be lattices in $\mathbb{R}^n$.
  Then L' is a *sublattice* of L if $L' \subseteq L$.
- **Definition**. A *full-rank lattice* L in $\mathbb{R}^n$ is a lattice in $\mathbb{R}^n$ of rank n.
- Henceforth, unless otherwise stated, all lattices and sublattices will be full-rank (and integer).
- Note that a basis $B = [b_1, b_2, \ldots, b_n]$ for a full-rank lattice in $\mathbb{R}^n$ is also a basis for the vector space $\mathbb{R}^n$.

Lattice: Example 1

- Let $n = 2$ and $B_1 = [(1,0),(0,1)]$.
- Then $L_1 = L(B_1) = \{B_1 x : x \in \mathbb{Z}^2\}$, where $B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Thus, $L_1 = \mathbb{Z}^n$.
- **Fundamental parallelepiped**:
  $P(B_1) = \{a_1(1,0) + a_2(0,1) : a_1, a_2 \in [0,1)\}$.



Fundamental parallelepiped

- **Definition**. Let $L = L(B)$ be a lattice in $\mathbb{R}^n$, where $B = [b_1, b_2, \ldots, b_n]$. The *fundamental parallelepiped* of L is $P(B) = \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n : a_i \in [0,1)\}$.
- **Notes**:

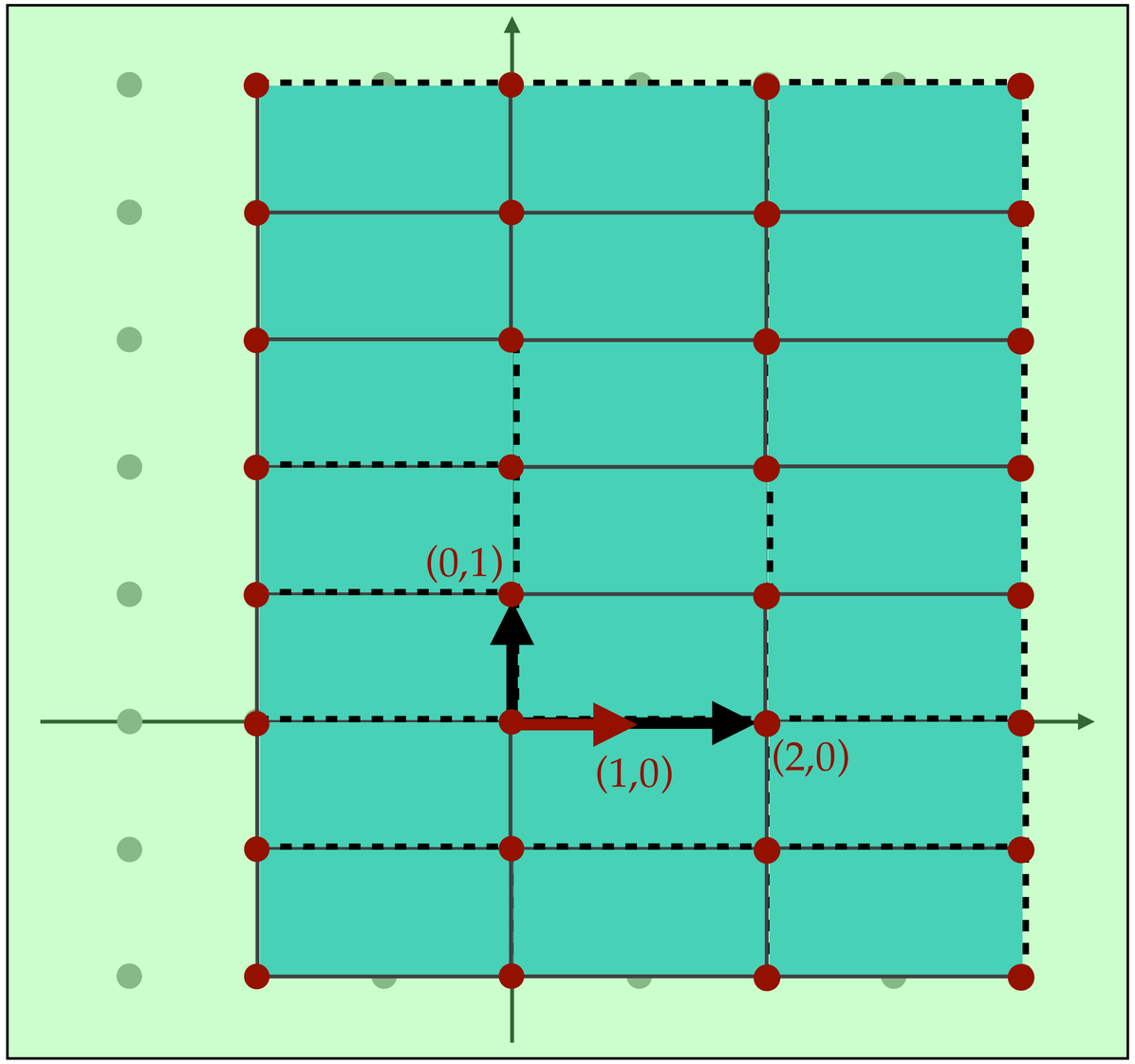
  1. Equivalently, $P(B) = \{Bx : x \in [0,1)^n\}$.
  2. $P(B)$ can be used to partition $\mathbb{R}^n$ into non-overlapping regions (called parallelepipeds). The "corners" of these parallelepipeds are the elements of the lattice $L(B)$.

Lattice: Example 2

- Let $n = 2$ and $B_2 = [(2,0), (0,1)]$.
- Then
  $L_2 = L(B_2) = \{B_2x : x \in \mathbb{Z}^2\}$,
  where $B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.
- **Notes**:

  1. L_2 a *sublattice* of L_1.

  2. $L_2 \neq L_1$ since $(1,0) \in L_1$, but
  $(1,0) = \frac{1}{2} \cdot (2,0) + 0 \cdot (0,1) \notin L_2$.



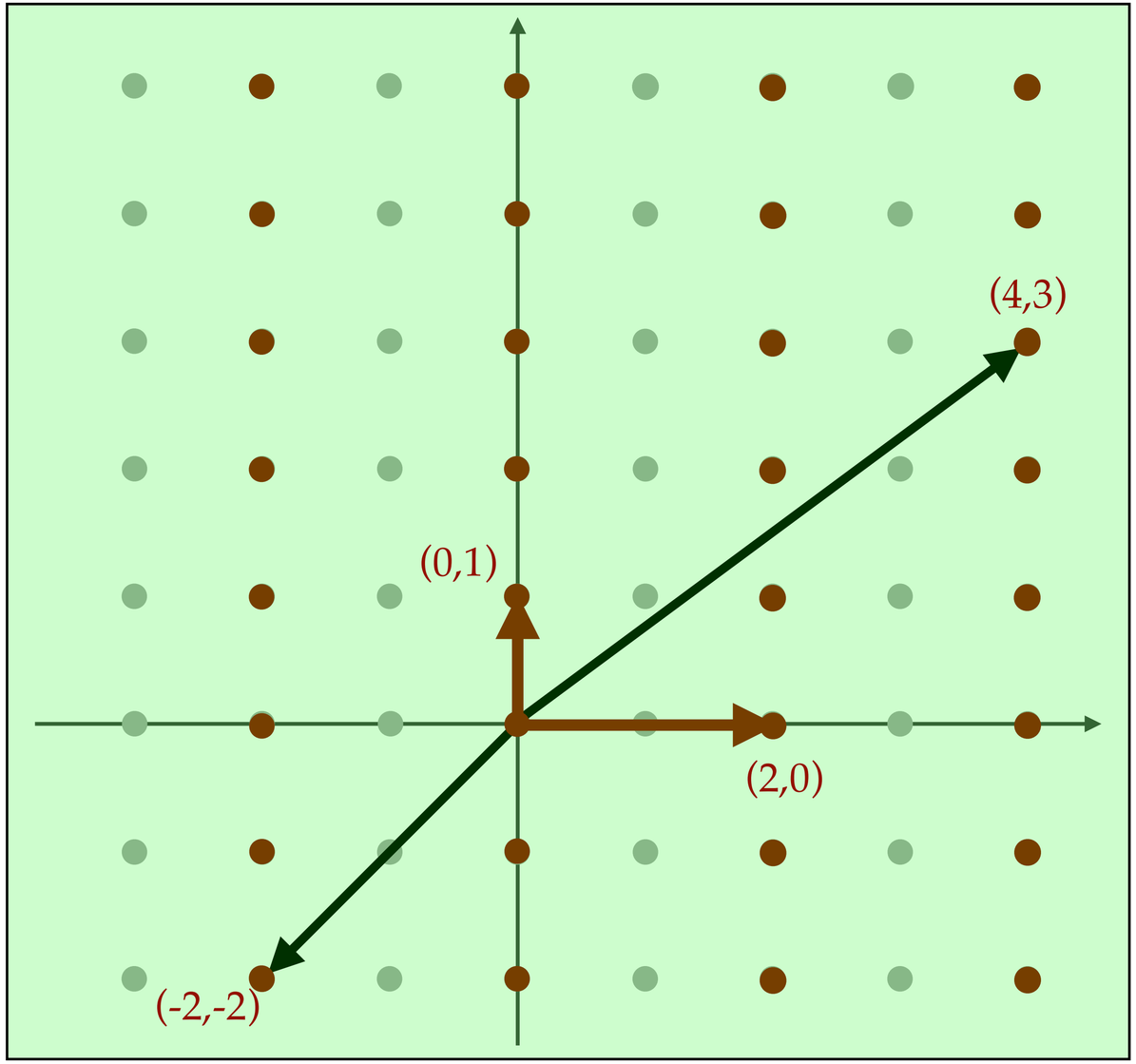
Lattice: Example 3

- Let $n = 2$ and $B_3 = [(-2,-2),(4,3)]$.
- Then $L_3 = L(B_3) = \{B_3 x : x \in \mathbb{Z}^2\}$, where $B_3 = \begin{bmatrix} -2 & 4 \\ -2 & 3 \end{bmatrix}$.
- **Notes**:

  1. $L_2 \subseteq L_3$ since
     $(2,0) = 3 \cdot (-2,-2) + 2 \cdot (4,3)$ and
     $(0,1) = -2 \cdot (-2,-2) - 1 \cdot (4,3)$.

  2. $L_3 \subseteq L_2$ since
     $(-2,-2) = -1 \cdot (2,0) - 2 \cdot (0,1)$ and
     $(4,3) = 2 \cdot (2,0) + 3 \cdot (0,1)$.

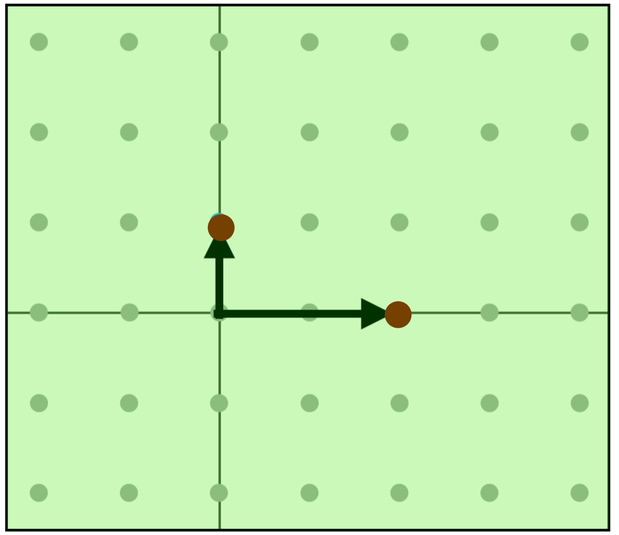
  3. Thus $L_3 = L_2$.



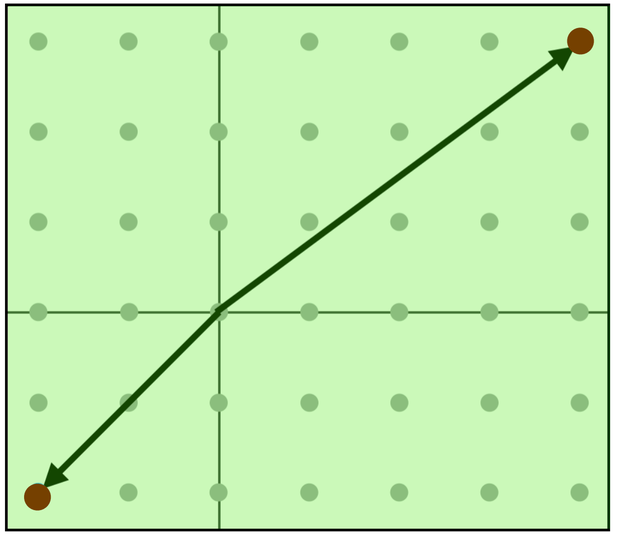
Some bases are "nicer" than others

- $L_2 = L([(2,0),(0,1)])$ and $L_3 = L([(-2,-2),(4,3)])$ are the same lattice, but described using different bases.
- The basis $B_2 = [(2,0),(0,1)]$ is "nicer" than the basis $B_3 = [(-2,-2),(4,3)]$ since the vectors in B_2 are "shorter" and "orthogonal" to each other.
- The *length* of a vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ is $\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ (also called the *Euclidean length* or ℓ_2- *norm*).

B_2

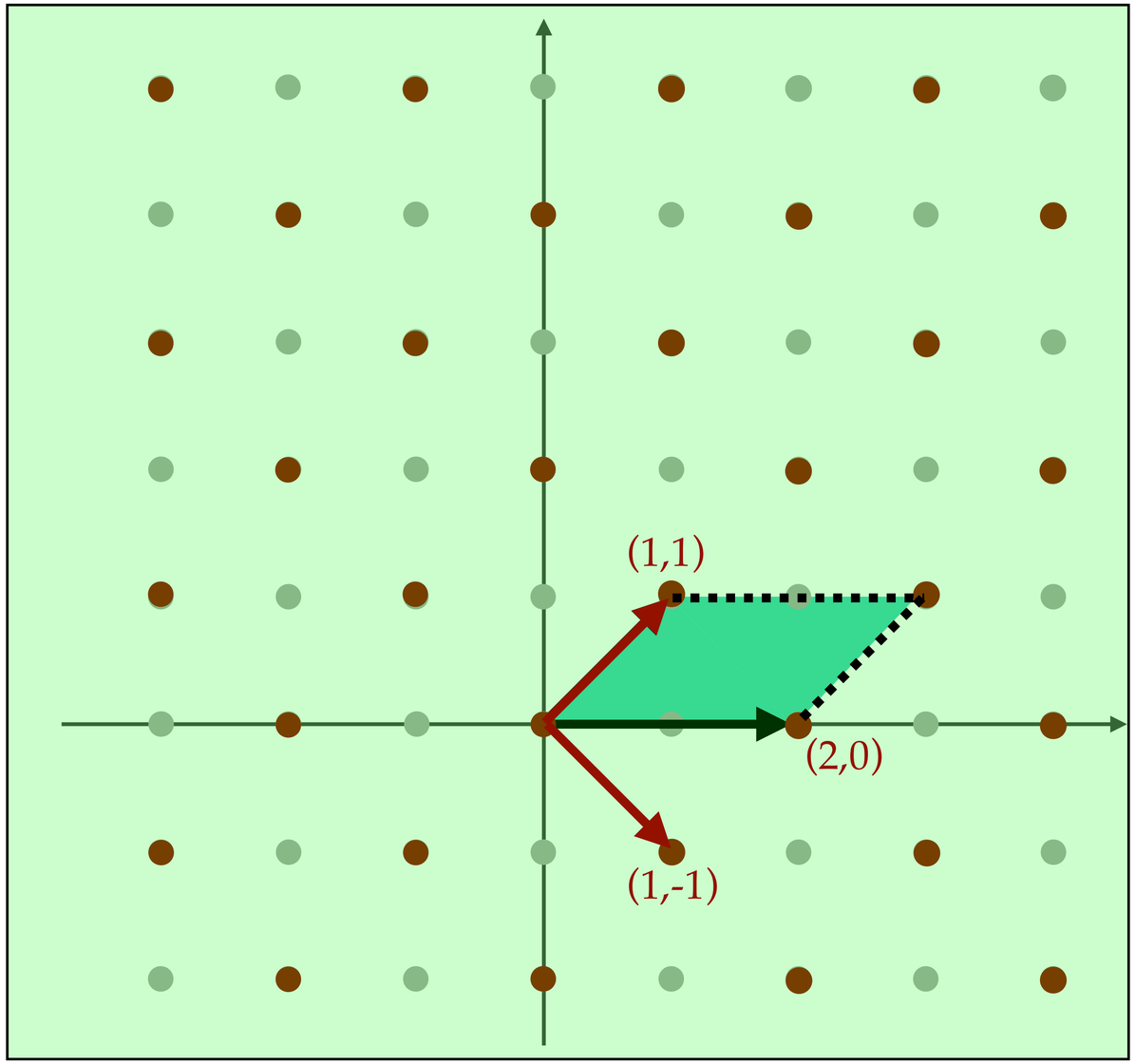


B_3



Lattice: Example 4

- Let $n = 2$ and $B_4 = [(2,0), (1,1)]$.
- Then $L_4 = L(B_4) = \{B_4 x : x \in \mathbb{Z}^2\}$, where $B_4 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.
- **Exercise:** Prove that $L_4 \neq L_1$ and $L_4 \neq L_2$.
- **Exercise:** Prove that $\{(1,-1), (1,1)\}$ is another (nicer) basis for L_4.



A lattice has infinitely many bases

- Let $L = L(B)$ be an n-dimensional (integer) lattice with (ordered) basis $B = [b_1, b_2, \ldots, b_n]$.
- **Exercise**: Show that the following operations yield a new basis B' for L:
  - Exchange two basis vectors: $b_i \leftrightarrow b_j$.
  - Multiply a basis vector by -1: $b_i \leftarrow -b_i$.
  - Add an integer multiple of a basis vector to another basis vector: $b_i \leftarrow b_i + cb_j$ where $c \in \mathbb{Z}$.
- It follows that the lattice L has infinitely many bases.

Characterization of lattice bases

- **Theorem** (*characterization of lattice bases*) Let $L = L(B_1)$ be an n-dimensional (integer) lattice. Then an $n \times n$ integer matrix B_2 is also a basis for L if and only if $B_1 = B_2 U$, where U is an $n \times n$ matrix (the change-of-basis matrix) with integer entries and with $\det(U) = \pm 1$.
  (Such a matrix U is called *unimodular*.)

- **Example.** $B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B_3 = \begin{bmatrix} -2 & 4 \\ -2 & 3 \end{bmatrix}$ are bases for the same lattice since

$$\underset{B_2}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{B_3}{\begin{bmatrix} -2 & 4 \\ -2 & 3 \end{bmatrix}} \cdot \underset{U}{\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}}$$

where U is a unimodular matrix.

Volume of a lattice

- **Definition**. Let $L = L(B)$ be a lattice. The volume of L is $\text{vol}(L) = |\det(B)|$.
- **Note:** The volume of a lattice is the "volume" of the fundamental parallelepiped of the lattice.
  - If the lattice is 2-dimensional, then its volume is the *area* of its parallelepiped.
  - Informally, the volume of a lattice is inversely proportional to the density of its lattice vectors. The larger the volume, the sparser is the lattice.
- **Exercise.** Show that the volume is an *invariant* of L, i.e., it doesn't depend on the basis B chosen for L.
- **Exercise**. Suppose that $L_1 \subseteq L_2$. Prove that $\text{vol}(L_1) \geq \text{vol}(L_2)$.

Successive minima

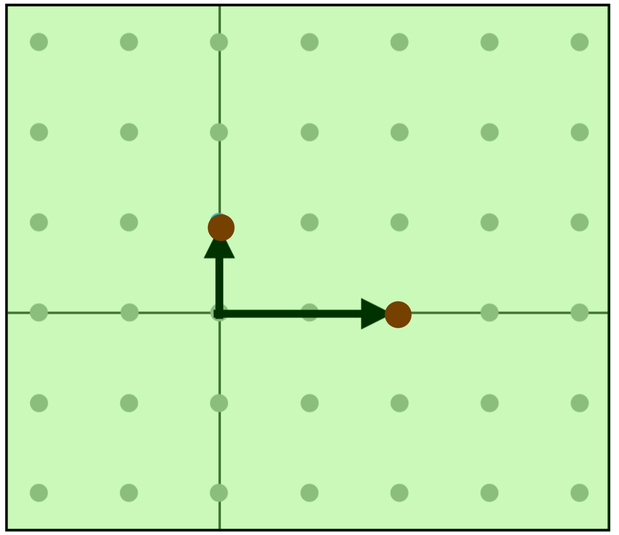
- A fundamental problem in lattice-based cryptanalysis is finding a "good" basis for a lattice.
- **Definition**: Let $L \subseteq \mathbb{Z}^n$ be a lattice. For each $i \in [1, n]$, the ith *successive minimum* $\lambda_i(L)$ is the smallest real number r such that L has i linearly independent vectors the longest of which has length r.
- **Notes**:

  1. $\lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L)$.
  2. $\lambda_1(L)$ is the length of a shortest nonzero vector in L.
  3. (*Minkowski's Theorem*) $\lambda_1(L) \leq \sqrt{n}\,\mathrm{vol}(L)^{1/n}$.
  4. (*Gaussian Heuristic*) $\lambda_1(L) \approx \sqrt{\frac{n}{2\pi e}}\,\mathrm{vol}(L)^{1/n}$ for random lattices.

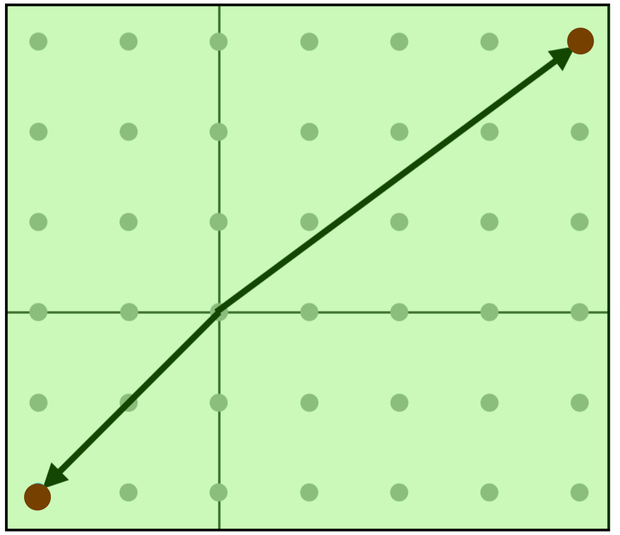
Some bases are "nicer" than others

- **Shortest Vector Problem (SVP)**: Given a lattice $L = L(B) \subseteq \mathbb{Z}^n$, find a shortest nonzero vector in L.
- **Example**: Consider the two SVP instances $L_2 = L([(2,0),(0,1)])$ and $L_3 = L([(-2,-2),(4,3)])$.
- So, hardness of an SVP instance $L(B)$ depends on the quality of the given basis B for L.

B_2



B_3



SVP: A fundamental lattice problem

- **Shortest Vector Problem (SVP)**: Given a lattice $L = L(B)$, find a lattice vector of length $\lambda_1(L)$.
  - SVP is **NP-hard**.
  - The fastest (classical) algorithm known for SVP has (heuristic) running time $2^{0.292n+o(n)}$.
  - The fastest quantum algorithm known for SVP has (heuristic) running time $2^{0.2563n+o(n)}$.
- **Approximate-SVP problem (SVP$_\gamma$)**: Given a lattice $L = L(B)$, find a nonzero lattice vector of length at most $\gamma \cdot \lambda_1(L)$.
  - SVP$_\gamma$ is believed to be hard for small γ.
    It's NP-hard for constant γ, but likely isn't NP-hard if $\gamma > \sqrt{n}$.
  - For $\gamma = 2^k$, the fastest algorithm known for SVP$_\gamma$ has running time $2^{\tilde{\Theta}(n/k)}$
    (where $\tilde{\Theta}$ hides a power of $\log n$).
  - If $\gamma > 2^{(n \log\log n)/\log n}$, then SVP$_\gamma$ can be efficiently solved using the LLL algorithm.

Low-rank lattices

- Let $L = L(B) \subseteq \mathbb{Z}^n$ be an n-dimensional lattice of rank m, where $B = [b_1, b_2, \ldots, b_m]$. (Here, $m \leq n$.)
- The volume of L is $\text{vol}(L) = \sqrt{\det(B^T B)}$.
  - Here, B is an $n \times m$ matrix, so $B^T B$ is an $m \times m$ matrix.
- **Note**: If $m = n$ (so L is a full-rank lattice), then $\text{vol}(L) = \sqrt{\det(B^T B)} = \sqrt{\det(B^T)\det(B)} = \sqrt{\det(B)^2} = |\det(B)|$.